On an implicit triangular decomposition of nonlinear control systems that are 1-flat - a constructive approach

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Abstract

We study the problem to provide an implicit triangular form for non-linear multi-input systems with respect to the flatness property. Furthermore, we suggest a constructive method, how to transform a given system into that special triangular shape, if possible. The well known Brunovsky form, which is applicable with regard to the input to state linearization problem, can be seen as special case of this implicit triangular form. A key tool in our investigation will be the construction of certain Cauchy characteristic vector fields that additionally annihilate certain distributions, and that will correspond to variables whose time-evolution can be derived without any integration.

Key words: Differential Flatness, Differential geometry, Pfaffian systems, Nonlinear control systems, Normal-forms

1 Introduction

The concept of flatness introduced in [6,7] has greatly influenced the control and systems theory community. The property of a system to be flat allows for an elegant solution for many feed-forward and/or feedback problems and is applicable for a big class of systems including the linear and the nonlinear as well as the lumped- and the distributed-parameter case. Within this paper we are interested in the system class of nonlinear multi-input systems described by ordinary differential equations. For this system class necessary and sufficient

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conditions for flatness have been proposed in [8,9] based on a polynomial matrix approach. Furthermore, for nonlinear multi-input systems with special structure further results exist, see e.g. [10,11,12,13].

Triangular forms are of special interest for nonlinear systems in the context of input to state linearizability or flatness. Systems that are input to state linearizable by static feedback can be converted to Brunovsky normal form (a special case of the extended Goursat form, adapted to control systems), see [5]. It is well known, that systems that are flat but not input to state linearizable by static feedback can be transformed into Brunovsky normal form only after a dynamic system extension (dynamic compensator), see [4]. For systems that are 0-flat in [2] a nonlinear explicit triangular form has been proposed. We consider systems that are 1-flat, the flat output may depend on the state (0-flat) and the control (1-flat) but not on the derivatives of the control. Our presented form relies on an *implicit triangular decomposition*, and we propose a constructive scheme how to transform a 1-flat system into that special form, if possible (this gives rise to a sufficient condition for a control system to be 1-flat). As in [5] where the extended Goursat form is discussed, we also make use of the Pfaffian system representation, where 1-forms (covector-fields) are used for the description of the (implicit) differential equations. Furthermore we will also make use of a filtration, which will lead to an implicit triangular representation in the case of flat systems. This is in contrast to the filtration which is based on *derived flags* used in the input to state linearization problem leading to an explicit representation like the Brunovsky form, see again [5]. The *implicit triangular decomposition* contains the Brunovsky form as a special case. A different approach based on differential forms, but associated with the tangent linear system can be found in [1], where the so-called infinitesimal Brunovsky form is considered.

This contribution can be seen as further developing the ideas presented in [14,15], where a reduction and elimination procedure is considered to derive flat outputs, but not based on a Pfaffian representation and the adequate tools from exterior algebra. Preliminary results have been presented already in [16], where also an extended example, the VTOL can be found, which has been analyzed using different tools e.g. in [7,8].

Notation: Let $\mathcal{X}$ be an n_x dimensional manifold equipped with local coordinates x^α, $\alpha = 1, \ldots, n_x$. We denote the partial derivatives by ∂_{x^α} and $\partial_x = \partial_{x^1}, \ldots, \partial_{x^\alpha}, \ldots, \partial_{x^{n_x}}$. We will make use of the Einstein convention on sums, namely $a_i b^i = \sum_{i=1}^{n} a_i b^i$ when the index range is clear from the context. If we have $\dot{x} = (\dot{x}^1, \ldots, \dot{x}^{n_x})$ as well as $\dot{x}^i = (\dot{x}^{i,1}, \ldots, \dot{x}^{i,n_{x^i}})$ an expression of the form $\Upsilon = \sum_{k=1}^{n_{x^i}} \sum_{i=1}^{n_x} a_{i,k}^j(x)\dot{x}^{i,k}$ reads according to Einstein's convention as $\Upsilon = a_{i,k}^j(x)\dot{x}^{i,k}$ where each $a_i(x)$ is a matrix of appropriate size and the indices j and k correspond to the rows and the columns of $a_i(x)$. In matrix

notation we have

$$a^j_{i,k}(x)\dot{x}^{i,k} = a_1(x)\dot{x}^1 + \ldots + a_{n_x}(x)\dot{x}^{n_x}$$

where each $\dot{x}^i$ is identified with a column vector.

2 The triangular form

Let us consider a nonlinear control system

$$\dot{x} = f(x, u) \tag{1}$$

with n_x states and n_u independent inputs. Roughly speaking the system (1) is flat ($(\kappa+1)-$flat), if there exist n_u differential independent functions $y(x, u, \dot{u}, \ldots, u^{(\kappa)})$, such that the state x and the control u can be parametrized by y and its successive time derivatives. The system is called 0-flat if y depends solely on x and 1-flat if we have $y(x, u)$. For a rigorous definition of differential flatness, see [6,8].

The Brunovsky-form, consisting of n_u Integrator chains is the most simple triangular structure that can be achieved for systems (1) that are input to state linearizable by static feedback, and hence 0-flat. A different (explicit) triangular form for 0-flat systems has been proposed in [2] which is more general than the Brunovsky form. To treat the case of 1-flat systems we will present an *implicit triangular form* for a special subclass of systems of the form (1), that is useful regarding property to be 1-flat.

Definition 1 *The implicit differential equations* $\Xi_{e,i} = 0\,,\ i = 1, \ldots, m$ *(affine in the derivative coordinates* $\dot{z}$*)*

$$\begin{aligned}
\Xi_{e,1} &: a^{1,j_1}_{1,\alpha_1}\dot{z}^{1,\alpha_1} - b^{1,j_1}\\
\Xi_{e,2} &: a^{2,j_2}_{1,\alpha_1}\dot{z}^{1,\alpha_1} + a^{2,j_2}_{2,\alpha_2}\dot{z}^{2,\alpha_2} - b^{2,j_2}\\
&\ \vdots\\
\Xi_{e,m} &: a^{m,j_m}_{i,\alpha_l}\dot{z}^{i,\alpha_l} - b^{m,j_m}
\end{aligned} \tag{2}$$

with $j_i = 1, \ldots \dim(\Xi_{e,i})$ *and* $\alpha_i = 1, \ldots, n_{z^i}$ *(*$i = 1, \ldots, m$*) are termed the implicit triangular form (*j_i *and* α_i *are indices corresponding to the rows and the columns of the matrices* a^i_l *and vectors* b^i*, respectively), which possesses the following properties*

(a) $z = (z^1, \ldots, z^i, \ldots, z^m)$*, where each* z^i *is of the form* $z^i = (z^{i,1}, \ldots, z^{i,n_{z^i}})$

(b) $z^1 = y^1$ *and we have a partition of the form*

$$z^2 = (y^2, \hat{z}^2), \ldots, z^m = (y^m, \hat{z}^m)$$

with $n_{z^i} = n_{y^i} + n_{\hat{z}^i}$ *and* $n_{\hat{z}^1} = 0$

(c) $n_{z^i} > 0\,,\ \ i = 1, \ldots, m\,,\quad n_{y^j} \geq 0\,,\ \ j = 2, \ldots, m$

(d) the matrices a_k^i *and (the vectors)* b^i *meet*

$$\begin{aligned} a^{i,j_l}_{k,\alpha_l} &= a^{i,j_l}_{k,\alpha_l}(z^1, \ldots, z^i, \hat{z}^{i+1})\,, \\ b^{i,j_l} &= b^{i,j_l}(z^1, \ldots, z^i, \hat{z}^{i+1}) \end{aligned} \tag{3}$$

(e) $\dim(\Xi_{e,i}) = \dim(\hat{z}^{i+1})$*, and the Jacobian matrices* $[\partial_{\hat{z}^{i+1}}\Xi_{e,i}]$ *are regular for all* $i = 1, \ldots, m$*.*

Example 2 *A system in triangular form with* $m = 3$ *in matrix notation reads as*

$$\begin{aligned} &\Xi_{e,1} : a_1^1(z^1, \hat{z}^2)\dot{z}^1 - b^1(z^1, \hat{z}^2) \\ &\Xi_{e,2} : a_1^2(z^1, z^2, \hat{z}^3)\dot{z}^1 + a_2^2(z^1, z^2, \hat{z}^3)\dot{z}^2 - b^2(z^1, z^2, \hat{z}^3) \\ &\Xi_{e,3} : a_i^3(z^1, z^2, z^3, \hat{z}^4)\dot{z}^i - b^3(z^1, z^2, z^3, \hat{z}^4) \end{aligned}$$

where a summation from $i = 1, \ldots, 3$ *takes place and*

$$z^1 = y^1\,,\ \ z^2 = (y^2, \hat{z}^2)\,,\ \ z^3 = (y^3, \hat{z}^3)$$

such that y^2 *and/or* y^3 *are possibly empty (but they need not,* $n_{y^2}, n_{y^3} \geq 0$ *). We require* $\dim(\Xi_{e,i}) = \dim(\hat{z}^{i+1})$*, and the Jacobian matrices* $[\partial_{\hat{z}^{i+1}}\Xi_{e,i}]$ *are regular for all* $i = 1, \ldots, 3$ *such that* $\hat{z}^{i+1}$*can be computed by means of the implicit function theorem from* $\Xi_{e,i}$*.*

Lemma 3 *y is a flat output for the system (2).*

To proof this Lemma, we consider the implicit equations $\Xi_{e,1} = 0$. Then $z^1(t) = y^1(t)$ can be chosen freely, and $\hat{z}^2(t)$ can be computed, where we make use of the implicit function theorem. We continue with the equations $\Xi_{e,2} = 0$. Two scenarios are possible: $z^2 = \hat{z}^2$, then it can be easily checked that $\hat{z}^3(t)$ can be computed using the same argument as for $\hat{z}^2(t)$, since $z^1(t)$ and $z^2(t)$ are already given. If $z^2 = (y^2, \hat{z}^2)$ then $y^2(t)$ can be chosen freely, since the rank criteria is met for $\hat{z}^3(t)$, which again can be computed. By continuing these procedure, we end up with the equations $\Xi_{e,m} = 0$ from which $\hat{z}^{m+1}(t)$ can be computed since at this stage $z^1(t), \ldots, z^m(t)$ are already known. This clearly shows that $y^1, \ldots, y^m$ is a flat output for (2).

Proposition 4 *(a sufficient condition) The control system (1) is 1-flat if we can find locally a diffeomorphism* $(x, u) = \varphi(z^1, \ldots, z^m, \hat{z}^{m+1})$ *with* $(\sum_{i=1}^m n_{z^i})+$

$n_{\hat{z}^{m+1}} = n_u + n_x$, such that it can be represented in the form (2) and $\sum_{i=1}^{m} n_{y^i} = n_u$.

The fact that this proposition is sufficient for 1-flat systems, comes from the observation that the flat outputs are among the z coordinates, and therefore clearly a function of x and u - as a special case also 0-flat systems are included. Furthermore it should be noted that we consider a diffeomorphism $(x, u) = \varphi(z^1, \ldots, z^m, \hat{z}^{m+1})$ which implies that we do not increase the dimension of the system variables.

The goal is now to provide a constructive algorithm that transforms a nonlinear multi-input system (1), if possible, into the form (2). Before we will analyze this in detail, let us consider an example.

2.1 A motivating example

We consider a system with 3 state variables (x^1, x^2, x^3) and 2 control inputs (u^1, u^2) of the form

$$\dot{x}^1 = u^1, \ \ \dot{x}^2 = u^2, \ \ \dot{x}^3 = \sin\left(\frac{u^1}{u^2}\right) \tag{4}$$

also analyzed in [8,14] using a different approach. Let us introduce the local coordinate transformation $(x^1, x^2, x^3, u^1, u^2) \leftrightarrow (y^1, \hat{z}^2, y^2, \hat{z}^3, \hat{z}^4)$ with

$$\begin{aligned}
x^1 &= \hat{z}^2 \hat{z}^3 & \qquad y^1 &= x^3 \\
x^2 &= \hat{z}^3 + y^2 & \qquad \hat{z}^2 &= \tfrac{u^1}{u^2} \\
x^3 &= y^1 & \qquad y^2 &= x^2 - x^1 \tfrac{u^2}{u^1} \\
u^1 &= e^{\hat{z}^4} \hat{z}^2 & \qquad \hat{z}^3 &= x^1 \tfrac{u^2}{u^1} \\
u^2 &= e^{\hat{z}^4} & \qquad \hat{z}^4 &= \ln(u^2).
\end{aligned} \tag{5}$$

Then the system (4) in the new coordinates can be represented as

$$\begin{aligned}
\dot{y}^1 & & & - \sin(\hat{z}^2) &= 0 \\
-\dot{y}^2 \hat{z}^2 + \dot{\hat{z}}^2 \hat{z}^3 & & & &= 0 \\
\dot{y}^2 & + \dot{\hat{z}}^3 & & - e^{\hat{z}^4} &= 0
\end{aligned} \tag{6}$$

(by a suitable combination of the equations) which is an implicit system of differential equations and the flat outputs are obviously y^1 and y^2. Following Lemma 3 this can be seen as follows: Assigning $y^1(t)$ we can compute $\hat{z}^2(t)$.

Then by choosing $y^2(t)$ we can compute $\hat{z}^3(t)$ from the second equation since $\hat{z}^2(t)$ is already given. And from the last equation $\hat{z}^4(t)$ follows since $\hat{z}^3(t)$ and $y^2(t)$ are already given. Therefore, it is easily verified that the system variables $(\hat{z}^2, \hat{z}^3, \hat{z}^4)$ are parametrized with respect to the flat outputs y^1, y^2 and its derivatives. Using (5) one obtains the system parametrization in the coordinates (x, u) and the flat outputs in original coordinates follow as $y^1 = x^3\,,\ \ y^2 = x^2 - x^1\frac{u^2}{u^1}$.

Example 5 *The system (6) possesses the structure (2) as in example 2 with* $m = 3$ *and* $z^1 = y^1\,,\ z^2 = (y^2, \hat{z}^2)\,,\ z^3 = \hat{z}^3$ *and the matrices*

$$
\begin{aligned}
&a_1^1 = 1\ , && b^1 = \sin(\hat{z}^2)\\
&a_1^2 = 0\ ,\ a_2^2 = [-\hat{z}^2, \hat{z}^3]\ , && b^2 = 0\\
&a_1^3 = 0\ ,\ a_2^3 = \begin{bmatrix} 1 & , & 0 \end{bmatrix}\ ,\ a_3^3 = 1\ , && b^3 = e^{\hat{z}^4}.
\end{aligned}
$$

The key question is now, how to derive the coordinate transformation (5) and how must the equations be combined, such that the form (6) can be obtained. These two questions will be answered at once by using a Pfaffian system representation.

3 Pfaffian representation

We will use some facts about exterior algebra and Pfaffian systems in the sequel. For detailed information see [3]. More information concerning control systems in a Pfaffian representation can be found e.g. in [5] and references therein. It should be noted that we do not base our considerations on the tangent linear system, as it is used for instance in [1].

3.1 Introduction to Exterior Algebra

We denote $\mathrm{d}\omega$ the exterior derivative of the k-form ω and by $v\rfloor\omega$ the contraction (interior product) of ω by the vector field v. The exterior product (wedge product) is denoted by $\wedge$.

A *Pfaffian system* I on a n_x-dimensional manifold $\mathcal{X}$ with coordinates x^α is identified with a codistribution on $\mathcal{X}$ and is locally spanned by 1-forms ω on $\mathcal{X}$.

The *derived flag* of the Pfaffian System I is the descending chain of Pfaffian

systems $I^{(0)} \supset I^{(1)} \supset I^{(2)} \supset \ldots$with $I^{(0)} = I$ and

$$I^{(k+1)} := \left\{ \omega \in I^{(k+1)}, \mathrm{d}\omega \equiv 0 \mod I^{(k)} \right\}.$$

Cauchy characteristic vector fields v of I denoted by $v \in \mathcal{C}(I)$ meet

$$(a)\ v \rfloor I = 0\,, \quad (b)\ v \rfloor (\mathrm{d}I) \subset I. \tag{7}$$

The importance of Cauchy characteristic vector fields lies in the fact, that the Pfaffian system I can be written using $n_x - c$ coordinates (after a suitable coordinate transformation), where c denotes the number of all independent Cauchy characteristic vector fields. Due to the fact that the distribution formed by all Cauchy characteristic vector fields is involutive, one can choose coordinates on $\mathcal{X}$ adapted to the vector fields, i.e. $(\bar{x}^1, \ldots, \bar{x}^c, \bar{\bar{x}}^{c+1}, \ldots \bar{\bar{x}}^{n_x})$, because of the Frobenius theorem, see [3], such that $\mathcal{C}(I) = \{\partial_{\bar{x}}\}$. Because of (a) in (7) the forms in I do not include differentials $\mathrm{d}\bar{x}$ and because of (b) in (7) one can chose a basis for I such that the coordinates $\bar{x}$ do not enter the function part of the forms, i.e. I can be expressed by the $n_x - c$ coordinates $\bar{\bar{x}}$, for the proof see again [3]. The same holds true for any involutive subdistribution $\bar{\mathcal{C}}(I) \subset \mathcal{C}(I)$ with $\dim(\bar{\mathcal{C}}(I)) = \bar{c} < c$ such that I can be expressed by the $n_x - \bar{c}$ coordinates.

The *annihilator* of a Pfaffian system I is a distribution denoted by $I^{\perp} := \left\{ w \in I^{\perp},\ w \rfloor \omega = 0,\ \forall \omega \in I \right\}$.

3.2 Properties of time-invariant implicit Systems in Pfaffian representation

To handle time-invariant implicit dynamical systems denoted by S we consider a fibration of a manifold $\mathcal{Z}$ over a one dimensional manifold (equipped with the time coordinate t), such that S is spanned by 1-forms reading as

$$\omega^i = m^i_\alpha(z)\mathrm{d}z^\alpha - n^i(z)\mathrm{d}t \tag{8}$$

on a manifold with dimension $n_z + 1$. To ω^i there correspond the implicit differential equations $\omega^i_e = 0$ with $\omega^i_e = m^i_\alpha(z)\dot{z}^\alpha - n^i(z)$.

Definition 6 *The vertical annihilator of (8) denoted by* $\mathcal{V}(S)^{\perp}$ *consist of vertical vector fields (tangential to the fibration)* $w = w^\alpha(z)\partial_{z^\alpha}$ *that annihilate* S*.*

Example 7 *Let us consider the control system* $\dot{x} = f(x,u)$ *written as a Pfaffian system* $S = \{\omega^\alpha\}$ *with*

$$\omega^\alpha = \mathrm{d}x^\alpha - f^\alpha(x,u)\mathrm{d}t.$$

Then we have $S^{\perp} = \{\partial_t + f^\alpha(x,u)\partial_{x^\alpha}, \partial_u\}$ *as well as* $\mathcal{V}(S)^{\perp} = \{\partial_u\}$*.*

If the Pfaffian system S as in (8) takes the following special form

$$\omega^i = m^i_\alpha(\bar{z}, \hat{z})\mathrm{d}\bar{z}^\alpha - n^i(\bar{z}, \hat{z})\mathrm{d}t \tag{9}$$

then the variables $\hat{z}$ are termed *non-derivative* variables. It should be noted that $\partial_{\hat{z}} \in \mathcal{V}(S)^\perp$ holds. If we furthermore assume that the implicit differential equations corresponding to (9), i.e. $\omega^i_e = 0$ with $\omega^i_e = m^i_\alpha(\bar{z}, \hat{z})\dot{\bar{z}}^\alpha - n^i(\bar{z}, \hat{z})$ fulfill that the Jacobian matrix $[\partial_{\hat{z}}\omega^i_e]$ is regular and $i = 1, \ldots, n_{\hat{z}}$ then we call the system S *parameterizable* with respect to $\hat{z}$ since then $\hat{z} = g(\bar{z}, \dot{\bar{z}})$, which follows at once by the implicit function theorem.

3.3 The implicit triangular form in Pfaffian representation

Let us consider the implicit differential equations as in (2) written using differential forms with the same properties as described in definition 1 (a) - (e) where Ξ_i corresponds to the Pfaffian representation of $\Xi_{e,i}$

$$\begin{aligned}
\Xi_1 &: a^{1,j_1}_{1,\alpha_1}\mathrm{d}z^{1,\alpha_1} - b^{1,j_1}\mathrm{d}t \\
\Xi_2 &: a^{2,j_2}_{1,\alpha_1}\mathrm{d}z^{1,\alpha_1} + a^{2,j_2}_{2,\alpha_2}\mathrm{d}z^{2,\alpha_2} - b^{2,j_2}\mathrm{d}t \\
&\vdots \\
\Xi_m &: a^{m,j_m}_{i,\alpha_l}\mathrm{d}z^{i,\alpha_l} - b^{m,j_m}\mathrm{d}t
\end{aligned} \tag{10}$$

Let us denote by[1] $S_{d,0}$ the system (10) and by $S_{d,k} = \{\Xi_1, \ldots, \Xi_{m-k}\}$.

Proposition 8 *The system (10) enjoys the following properties*

(a) $\mathcal{V}(S_{d,k})^\perp = \{\partial_{\hat{z}^{m+1-k}}\}$ are involutive distributions and $\partial_{\hat{z}^{m+1-k}} \in \mathcal{C}(S_{d,k+1})$ for $k = 0, \ldots, m-1$.

(b) Each subsystem Ξ_k is parameterizable with respect to the non-derivative variable $\hat{z}^{k+1}$, i.e.

$$\hat{z}^{k+1} = g^{k+1}(z^1, \ldots, z^k, \dot{z}^1, \ldots, \dot{z}^k)$$

for $k = 1, \ldots, m$.

(c) If in Ξ_k, $z^k = (y^k, \hat{z}^k)$ such that $n_{z^k} > n_{\hat{z}^k}$, i.e. variables y^k are present, then, $\partial_{y^k} \in \mathcal{C}(S_{d,m-k+1})$.

The proof of this proposition is straightforward and follows from the structure of (10) together with the special structure of the a^{i,j_l}_{k,α_l} and b^{i,j_l} according to (3) as in definition 1.

[1] The subscript d will always refer to a representation based on the desired triangular decomposition (10).

Corollary 9 *The implicit triangular decomposition (10) gives rise to the decomposition of $S_{d,0}$ into a nested sequence*

$$\ldots \subset S_{d,2} \subset S_{d,1} \subset S_{d,0}$$

as well as to splittings of the form $S_{d,i} = S_{d,i+1} \oplus S_{d,i+1,c}$ where all the $S_{d,i+1,c}$ are parameterizable, with respect to the corresponding non-derivative variables $\hat{z}$.

Example 10 *(Example 2 cont.) Following the notations in proposition 8 we have $S_{d,0} = \{\Xi_1, \Xi_2, \Xi_3\}$, $S_{d,1} = \{\Xi_1, \Xi_2\}$ and $S_{d,2} = \{\Xi_1\}$ since $m = 3$. From (a) we observe that $\mathcal{V}(S_{d,0})^\perp = \{\partial_{\hat{z}^4}\} \subset \mathcal{C}(S_{d,1})$ which is obvious since $\hat{z}^4$ is a non-derivative variable which only appears in Ξ_3. The same holds true regarding $\hat{z}^3$ where now $\mathcal{V}(S_{d,1})^\perp = \{\partial_{\hat{z}^3}\} \subset \mathcal{C}(S_{d,2})$. Proposition 8 (c) means for instance that if $z^2 = (y^2, \hat{z}^2)$ then $\partial_{y^2} \in \mathcal{C}(S_{d,2})$ since in Ξ_1 only $\hat{z}^2$ appears.*

4 A constructive algorithm

The goal is now to develop a constructive scheme that subsequently creates this nested sequence based on a given control system of the form $S_0 = \{\omega_0^\alpha\}$

$$\omega_0^\alpha = \mathrm{d}x^\alpha - f^\alpha(x, u)\mathrm{d}t. \tag{11}$$

The starting point of the scheme is the explicit system S_0 but since linear combinations of the ω_0^α lead to implicit equations in general we demonstrate the constructive method with the system $S_k = \{\omega_k^i\}$ (here the index k refers to the $k-th$ iteration of the reduction process) with

$$\omega_k^i = m_\alpha^i(\bar{x})\mathrm{d}\bar{x}^\alpha - n^i(\bar{x})\mathrm{d}t \tag{12}$$

with $i = 1, \ldots, n_e$ and $n_{\bar{x}} > n_e$, where we denote by $\bar{x}$ all the system variables. (11) is a special case of (12), i.e. $\bar{x} = (x, u)$ in S_0. The following steps need to be performed

(a) Computation of $\mathcal{V}(S_k)^\perp$, since these elements correspond to non-derivative variables. Choosing of an involutive $\mathcal{F}_k \subset \mathcal{V}(S_k)^\perp$ corresponds to a selection of non-derivative variables called $\hat{w}_k$. (This correspondence becomes obvious in an adapted coordinate chart to be constructed by means of the flow box theorem.)

(b) Construction of a splitting $S_k = S_{k+1} \oplus S_{k+1,c}$ such that $\mathcal{F}_k \subset \mathcal{C}(S_{k+1})$, since this guarantees that S_{k+1} is independent of $\hat{w}_k$.

(c) Check, if $S_{k+1,c}$ is parameterizable with respect to the $\hat{w}_k$, which is possible only if $\dim(S_k) = \dim(S_{k+1}) + \dim(\mathcal{F}_k)$ holds.

The whole procedure will then be continued with S_{k+1}.

The constructive scheme rests on the following proposition

Proposition 11 *Let us consider the system $S_k = \{\omega_k^i\}$ with ω_k^i as in (12). If we find an involutive distribution $\mathcal{F}_k$ with $\mathcal{F}_k \in \mathcal{V}(S_k)^\perp$ and a sub-codistribution $S_{k+1} \subset S_k$ such that $\mathcal{F}_k \in \mathcal{C}(S_{k+1})$ is met, then we obtain a splitting $S_k = S_{k+1} \oplus S_{k+1,c}$ with*

$$\begin{aligned} S_{k+1} &: \omega_{k+1}^i = a_\alpha^i(w_k)\mathrm{d}{w_k}^\alpha - b^i(w_k)\mathrm{d}t \\ S_{k+1,c} &: \omega_{k+1,c}^j = a_{\alpha,c}^j(w_k, \hat{w}_k)\mathrm{d}{w_k}^\alpha - b_c^j(w_k, \hat{w}_k)\mathrm{d}t \end{aligned} \tag{13}$$

in adapted coordinates $(w_k, \hat{w}_k)$ by using a diffeomorphism $\bar{x} = \varphi_k(w_k, \hat{w}_k)$ with $n_{\bar{x}} = n_{w_k} + n_{\hat{w}_k}$.

To prove this proposition we consider the case $\dim(\mathcal{F}_k) = 1$ first. A distribution generated by a single vector field v_k is involutive by construction, and one performs a coordinate change based on the flow of v_k which is denoted by $\phi_{\hat{w}_k}$ with the flow parameter $\hat{w}_k$. If we change coordinates according to

$$\bar{x} = \phi_{\hat{w}_k} \circ \psi(w_k) = \varphi_k(\hat{w}_k, w_k)\,,\ n_{\bar{x}} = n_{w_k} + n_{\hat{w}_k}\,,\ n_{\hat{w}_k} = 1$$

where ψ is chosen such that φ_k is a local diffeomorphism, we obviously obtain in new coordinates $v_k \to \partial_{\hat{w}_k}$ and it can be checked, that $\partial_{\hat{w}_k} \subset \mathcal{V}(S_k)^\perp$ as well as $\partial_{\hat{w}_k} \subset \mathcal{C}(S_{k+1})$ is met, therefore no $\mathrm{d}\hat{w}_k$ can appear and a basis of S_{k+1} must exist which is independent of the $\hat{w}_k$ coordinates, since $\partial_{\hat{w}_k} \subset \mathcal{C}(S_{k+1})$.

Remark 12 *The flow of $\phi_{\hat{w}_k}$ is a map $\bar{x} = \phi_{\hat{w}_k}(\tilde{x})$, where $\tilde{x}$ corresponds to the initial conditions. Since the flow parameter $\hat{w}_k$ is part of the new coordinates, and we require a diffeomorphism, the map $\tilde{x} = \psi(w_k)$ is at need, which cuts the flow $\phi_{\hat{w}_k}$ transversal.*

To generalize this to $\dim(\mathcal{F}_k) > 1$ is straightforward, since then the coordinate transformation is based on the composition of the flows generated by the fields of $\mathcal{F}_k$ (where the pairwise Lie-brackets of these fields have to vanish).

Based on these considerations we state the following corollary which additionally includes the parametrization criteria, such that proposition (11) is connected with the triangular form (2), respectively (10).

Corollary 13 *The system S_0 (11) can be transformed into the form (10) if we find a nested sequence of codistributions*

$$\ldots \subset S_2 \subset S_1 \subset S_0$$

as well as involutive distributions $\mathcal{F}_l$ with $\dim(\mathcal{F}_l) = l_r$ that meet $\mathcal{F}_l \in \mathcal{V}(S_l)^\perp$ as well as $\mathcal{F}_l \in \mathcal{C}(S_{l+1})$ for $l \geq 0$ such that the systems $S_{l+1,c}$ according to $S_l = S_{l+1} \oplus S_{l+1,c}$ are parameterizable. Then also $\dim(S_l) = \dim(S_{l+1}) + l_r$ holds.

This sequence ends when we have a decomposition of the form $S_{k^*} = S_{k^*+1} \oplus S_{k^*+1,c}$ with S_{k^*+1} the empty system, which means that S_{k^*} is a parameterizable system. This iterative scheme has therefore to be continued until a parameterizable system is obtained. It should be noted that in practice the effective computation of $\mathcal{F}_l$ and S_{l+1} such that additionally parametrization is guaranteed for all elements of the sequence is a difficult task. We will comment on computational issues in the forthcoming and demonstrate on an example a possible strategy.

4.2 The connection with the derived flag

In this short paragraph we want to discuss how the derived flag, see [3,5] is connected to our filtration, as in corollary 13.

Let us introduce an adapted basis for S_k, $\dim(S_k) = h$ (with respect to the derived flag) which is $S_k = \{\bar{\Omega}_k^1, \ldots, \bar{\Omega}_k^j, \Omega_k^{j+1}, \ldots, \Omega_k^h\}$. The first derived flag of S_k, denoted by $S_k^{(1)}$, meets $\bar{\Omega}_k^j \in S_k^{(1)} \subseteq S_k$, $j = 1, \ldots, \dim(S_k^{(1)})$ such that

$$\mathrm{d}\bar{\Omega}_k^j = \alpha_l^j \wedge \bar{\Omega}_k^l + \beta_r^j \wedge \Omega_k^r \tag{14}$$

holds for suitable 1-forms α_l^j, β_r^j.

If $\{S_k^{(1)}, \mathrm{d}t\}$ is integrable (Frobenius theorem, [3]), we have furthermore that

$$\mathrm{d}\bar{\Omega}_k^j = \gamma_l^j \wedge \bar{\Omega}_k^l + \rho^j \wedge \mathrm{d}t \tag{15}$$

is met, for suitable 1-forms γ_l^j, ρ^j.

Proposition 14 *Let us consider any sequence of Pfaffian systems $\ldots \subset S_2 \subset S_1 \subset S_0$ with S_0 as in (11) with elements S_k together with their first derived systems $S_k^{(1)}$. Then for all $v \in \mathcal{V}(S_k)^\perp$ we have*

(a) $v \rfloor \mathrm{d}S_k^{(1)} \subset S_k^{(1)}$ is met if $\{S_k^{(1)}, \mathrm{d}t\}$ is integrable.
(b) $v \rfloor (\mathrm{d}S_k^{(1)}) \subset S_k$

The proof of the first claim (a) follows by evaluating $v \rfloor \mathrm{d}\bar{\Omega}_k^j$ using (14) and (15) for $v \in \mathcal{V}(S_k)^\perp$ and the second (b) can be shown by using (14) in a straightforward manner.

Systems that are input to state linearizable by static feedback meet $\{S_k^{(1)}, \mathrm{d}t\}$ is integrable for every k, see [5].

Corollary 15 *From Proposition 14 it follows that for systems S_0 that are input to state linearizable by static feedback the sequence of the derived flags, corresponds to the sequence as in Corollary 13 and $\mathcal{F}_k$ corresponds to $\mathcal{V}(S_k)^\perp$ which is integrable by construction.*

Remark 16 *The interesting case are of course examples that are not input to state linearizable by static feedback, i.e. $\{S_k^{(1)}, \mathrm{d}t\}$ are not integrable, since then a different filtration has to be considered that may lead to an implicit triangular form.*

4.3 A constructive method to derive $\mathcal{F}_k$ and S_{k+1}

If one is able to construct the sequence as in Corollary 13, then one eventually end up with the form (10) (by relabeling the coordinates) where in each step the involutive distribution $\mathcal{F}_l$ has to be integrated, in order to derive the coordinate transformation. Thus, in principle a constructive method that generates the implicit triangular decomposition is stated. If then the rank and dimension condition as in corollary 13 hold the system is 0-flat/1-flat, but it should be stressed, that this method is only sufficient for flatness, and a failure does not in general proof that a system is not flat.

However, the construction of $S_{k+1} \subset S_k$ such that an involutive distribution $\mathcal{F}_k$ can be found that meets $\mathcal{F}_k \in \mathcal{V}(S_k)^\perp$ as well as $\mathcal{F}_k \in \mathcal{C}(S_{k+1})$ is a difficult task and lead in general to partial differential equations. Furthermore, since the choice $\mathcal{F}_k \in \mathcal{V}(S_k)^\perp$ as well as of S_{k+1} with $\mathcal{F}_k \in \mathcal{C}(S_{k+1})$ is not unique in general (may lead to an 'dead end' of the algorithm) it might be necessary to iterate the construction of $\mathcal{F}_k$ and S_{k+1} (see section 5.2) - it should be noted that based on a simple necessary condition candidates for $\mathcal{F}_k$ and S_{k+1} are singled out as shown next.

We have to construct $\mathcal{F}_k \in \mathcal{V}(S_k)^\perp$ and $S_{k+1} \subset S_k$ such that $\mathcal{F}_k \rfloor \mathrm{d}S_{k+1} \subset S_{k+1}$ is met. Then also the necessary condition

$$\mathcal{F}_k \rfloor \mathrm{d}S_{k+1} \subset S_k \tag{16}$$

holds, since $S_{k+1} \subset S_k$. For $\mathcal{V}(S_k)^\perp = \{v_i\}$ and $S_k = \{\omega^j\}$ with $r = \dim(S_k)$ we derive the purely algebraic conditions (necessary conditions)

$$c^i v_i \rfloor \mathrm{d}(a_j \omega^j) \wedge (\omega^1 \wedge, \ldots, \wedge \omega^r) = 0 \tag{17}$$

where c^i and a_j depend on all the system variables, i.e. $c^i(\bar{x})$ and $a_j(\bar{x})$ when considering the system (12).

It should be noted that due to the requirement $\dim(S_k) = \dim(S_{k+1}) + \dim(\mathcal{F}_k)$ one has to find $\dim(S_k) - \dim(\mathcal{F}_k)$ independent solutions $a_j \omega^j$ for S_{k+1}. From (b) in Proposition 14 we see that $S_k^{(1)}$ fulfills this necessary condition independently of $\mathcal{F}_k$. If we furthermore assume that $S_k^{(1)} \subset S_{k+1}$ then the construction of S_{k+1} and $\mathcal{F}_k$ can be simplified further as demonstrated in the next section in great detail. The strategy is now to solve the necessary conditions (16) or which is the same (17) and to generate solutions for which then finally the criteria $\mathcal{F}_k \rfloor \mathrm{d}S_{k+1} \subset S_{k+1}$ has to be checked as well as the parametrization as in corollary 13 .

5 Examples

We now present two examples, in the first one we show how one algorithmically can compute the nested sequence of codistributions using the necessary condition (17) and the second example demonstrates a case where the algorithm stops in a dead end, and another iteration is at need.

5.1 The motivating example revisited

Let us write the equations (4) as a Pfaffian system of the form $S_0 = \{\omega_0^1, \omega_0^2, \omega_0^3\}$

$$\begin{aligned} \omega_0^1 &= \mathrm{d}x^1 - u^1 \mathrm{d}t \\ \omega_0^2 &= \mathrm{d}x^2 - u^2 \mathrm{d}t \\ \omega_0^3 &= \mathrm{d}x^3 - \sin\left(\frac{u^1}{u^2}\right) \mathrm{d}t \end{aligned} \tag{18}$$

then we obtain the following proposition regarding the first reduction step

Proposition 17 *Given the system S_0 as in (18) we derive a splitting of the form $S_0 = S_1 \oplus S_{1,c}$ as well as v_0 that meets $v_0 \in \mathcal{V}(S_0)^{\perp}$ and $v_0 \in \mathcal{C}(S_1)$with*

$$v_0 = u^1 \partial_{u^1} + u^2 \partial_{u^2}$$

and $S_1 = \{\omega_1^1 = \omega_0^3, \omega_1^2 = u^2 \omega_0^1 - u^1 \omega_0^2\}$

$$\omega_1^1 = \mathrm{d}x^3 - \sin\left(\frac{u^1}{u^2}\right) \mathrm{d}t\,, \;\; \omega_1^2 = u^2 \mathrm{d}x^1 - u^1 \mathrm{d}x^2 \tag{19}$$

as well as a complement $S_{1,c} = \{\omega_{1,c}^3 = \omega_0^2 = \mathrm{d}x^2 - u^2 \mathrm{d}t\}$.

The proof of this proposition follows from the observation that $\mathcal{V}(S_0)^\perp = \{\partial_{u^1}, \partial_{u^2}\}$ and that $v_0 \rfloor \mathrm{d}S_1 \subset S_1$ as desired. We will now show how one can derive $S_1$and v_0.

Calculation 18 *The first derived system $S_0^{(1)}$ is given by the single form*

$$\Phi = \cos(\frac{u^1}{u^2})\left(\frac{u^1}{u^2}\mathrm{d}x^2 - \mathrm{d}x^1\right) + \mathrm{u}^2(\mathrm{d}x^3 - \sin(\frac{u^1}{u^2})\mathrm{d}t)$$

and a basis for S_0 can be alternatively given as $S_0 = \{\omega_0^1, \omega_0^2, \Phi\}$.

To construct S_1 we assume that $S_0^{(1)} \subset S_1$ and consider the relation (according to (17))

$$(c_1^1\partial_{u^1} + c_1^2\partial_{u^2})\rfloor \mathrm{d}(a_1^1\omega_0^1 + \mathrm{a}_2^1\omega_0^2 + a_3^1\Phi) \wedge \omega_0^1 \wedge \omega_0^2 \wedge \Phi = 0 \tag{20}$$

where c_1^i and a_i^1 are functions of all system variables, that have to be computed. From (20) we are left with the equation $c_1^1 a_1^1 + c_1^2 a_2^1 = 0$ or $a_1^1 = -a_2^1\frac{c_1^2}{c_1^1}$. This means that the forms Φ and $\omega_0^2 - \frac{c_1^2}{c_1^1}\omega_0^1$ fulfill the necessary conditions for the vector field $v_0 = c_1^1\partial_{u^1} + c_1^2\partial_{u^2}$. To determine c_1^1 and c_1^2 we consider the criteria $v_i\rfloor(\mathrm{d}S_1) \subset S_1$ and we derive the relation

$$\left((c_1^1\partial_{u^1} + c_1^2\partial_{u^2})\rfloor(\mathrm{d}\Phi)\right) \wedge (\omega_0^2 - \frac{c_1^2}{c_1^1}\omega_0^1) \wedge \Phi = 0. \tag{21}$$

For the solution of (21) of the form $c_1^1 = c_1^2\frac{u^1}{u^2}$ we have that Φ and $\omega_0^2 - \frac{u^2}{u^1}\omega_0^1$ clearly correspond to S_1 as in (19) as can be checked easily (by linear combinations) and that

$$(v_0\rfloor \mathrm{d}\omega_1^2) \wedge \omega_1^1 \wedge \omega_1^2 = 0 \tag{22}$$

is fulfilled, such that $v_0\rfloor(\mathrm{d}S_1) \subset S_1$ is met, because of (21,22).

The flow of v_0, is given as $(x, u) = \phi_{\hat{w}}(\tilde{x}, \tilde{u})$ with $x^i = \tilde{x}^i$ for $i = 1, 2, 3$ and

$$u^1 = e^{\hat{w}}\tilde{u}^4$$
$$u^2 = e^{\hat{w}}\tilde{u}^5$$

with flow parameter $\hat{w}$. We set $\tilde{x}^i = w^i$ for $i = 1, 2, 3$, $\tilde{u}^4 = w^4$ and $\tilde{u}^5 = 1$ (this corresponds the choice of ψ, i.e. $(\tilde{x}, \tilde{u} = \psi(w))$ to obtain locally the diffeomorphism $(x^1, x^2, x^3, u^1, u^2) = \varphi_0(w^1, w^2, w^3, w^4, \hat{w})$ and we obtain a basis for S_1 as

$$\begin{aligned} \omega_1^1 &: \mathrm{d}w^3 - \sin(w^4)\,\mathrm{d}t \\ \omega_1^2 &: \quad \mathrm{d}w^1 - w^4\mathrm{d}w^2 \end{aligned} \tag{23}$$

and for complement $S_{1,c}$

$$\omega^3_{1,c} : \mathrm{d}w^2 - e^{\hat{w}}\mathrm{d}t,$$

and it can be checked easily that $\dim(\mathcal{F}_0) = 1$, $\dim(S_1) + 1 = \dim(S_0)$ and that the Jacobian $\partial_{\hat{w}}(\dot{w}_2 - e^{\hat{w}})$ has maximal rank.

Remark 19 *We want to point out again, that v_0 is a Cauchy characteristic vector field for S_1, i.e. $v_0 \in \mathcal{C}(S_1)$ and this guarantees that there is a basis for S_1 which does not depend on the coordinate $\hat{w}$, since in new coordinates $\partial_{\hat{w}} \in \mathcal{C}(S_1)$ is met.*

Then we continue our considerations with S_1 and the following proposition states the second reduction step

Proposition 20 *Given the system S_1 as in (23) we derive a splitting of the form $S_1 = S_2 \oplus S_{2,c}$ as well as v_1 that meets $v_1 \in \mathcal{V}(S_1)^{\perp}$ and $v_1 \in \mathcal{C}(S_2)$ with*

$$v_1 = w^4\partial_{w^1} + \partial_{w^2} \tag{24}$$

and

$$S_2 = \left\{\omega^1_2 = \mathrm{d}w^3 - \sin\left(w^4\right)\mathrm{d}t\right\}, \tag{25}$$

and $S_{2,c} = \{\omega^2_{2,c} : \mathrm{d}w^1 - w^4\mathrm{d}w^2\}$

The proof follows again from the fact that $\mathcal{V}(S_1)^{\perp} = \{w^4\partial_{w^1} + \partial_{w^2}, \partial_{w^4}\}$ and $v_1\rfloor\mathrm{d}S_2 \subset S_2$. The construction of S_2 can be performed in the same manner as above. (Observe however that $S_1^{(1)}$ is empty, but from

$$(c^1_2(w^4\partial_{w^1} + \partial_{w^2}) + c^2_2\partial_{w^4})\rfloor\mathrm{d}(a^2_1\omega^1_1 + \mathrm{a}^2_2\omega^2_1) \wedge \omega^1_1 \wedge \omega^2_1 = 0$$

that result follows at once.).

Based on the flow of v_1 we derive the map

$$\begin{aligned} w^1 &= \hat{q}q^4 + q^1 \\ w^2 &= \hat{q} + q^2 \\ w^3 &= q^3 \\ w^4 &= q^4 \end{aligned}$$

with flow parameter $\hat{q}$ and we make the choice $q^1 = 0$ (this corresponds again the choice of ψ) leading to the transformation $w = \varphi_1(q, \hat{q})$. With

$$(y^1 = q^3,\, y^2 = q^2,\, \hat{z}^2 = q^4,\, \hat{z}^3 = \hat{q},\, \hat{z}^4 = \hat{w}) \tag{26}$$

it is easily seen that the composition of $(x,u)=\varphi_0(w,\hat{w})$ and $w=\varphi_1(q,\hat{q})$ together with (26) gives the desired transformation (5). Furthermore the nested sequence of systems $S_2 \subset S_1 \subset S_0$ leads at once to the desired normal-form

$$
\begin{aligned}
\omega_{d,0}^1 &= \mathrm{d}y^1 - \sin\left(\hat{z}^2\right)\mathrm{d}t \\
\omega_{d,0}^2 &= \hat{z}^3\mathrm{d}\hat{z}^2 - \hat{z}^2\mathrm{d}y^2 \\
\omega_{d,0}^3 &= \mathrm{d}\hat{z}^3 + \mathrm{d}y^2 - e^{\hat{z}^4}\mathrm{d}t.
\end{aligned}
$$

as in (6). The flow parameters $\hat{w}$ and $\hat{q}$ correspond to the non-derivative variables $\hat{z}^4$ and $\hat{z}^3$, respectively.

5.2 A further example

Let us consider the system $S_0=\{\omega_0^1,\omega_0^2,\omega_0^3,\omega_0^4\}$ also treated in [4] in a different context

$$
\begin{aligned}
\omega_0^1 &= \mathrm{d}x^1 - (x^2 + x^3u^2)\mathrm{d}t \\
\omega_0^2 &= \mathrm{d}x^2 - (x^3 + x^1u^2)\mathrm{d}t \\
\omega_0^3 &= \mathrm{d}x^3 - (u^1 + x^2u^2)\mathrm{d}t \\
\omega_0^4 &= \mathrm{d}x^4 - u^2\mathrm{d}t
\end{aligned}
$$

where we again have $\mathcal{V}(S_0)^\perp = \{\partial_{u^1}, \partial_{u^2}\}$.

The triangular form is based on the decompositions $S_0 = S_1 \oplus S_{1,c}$ with $S_1=\{\omega_1^1=\omega_0^1, \omega_1^2=\omega_0^2, \omega_1^3=\omega_0^4\}$

$$
\begin{aligned}
\omega_1^1 &= \mathrm{d}x^1 - (x^2 + x^3u^2)\mathrm{d}t \\
\omega_1^2 &= \mathrm{d}x^2 - (x^3 + x^1u^2)\mathrm{d}t \\
\omega_1^3 &= \mathrm{d}x^4 - u^2\mathrm{d}t
\end{aligned}
$$

and $S_{1,c}=\{\omega_{1,c}^4=\mathrm{d}x^3-(u^1+x^2u^2)\mathrm{d}t\}$ as well as on $S_1=S_2\oplus S_{2,c}$ with $S_2=\{\omega_2^1=\omega_1^1-u^2\omega_1^2,\ \omega_2^2=\omega_1^3\}$ with

$$
\begin{aligned}
\omega_2^1 &= \mathrm{d}x^1 - u^2\mathrm{d}x^2 - (x^2 - x^1(u^2)^2)\mathrm{d}t. \\
\omega_2^2 &= \mathrm{d}x^4 - u^2\mathrm{d}t
\end{aligned}
$$

and $S_{2,c}=\{\omega_{2,c}^3=\mathrm{d}x^2-(x^3+x^1u^2)\mathrm{d}t\}$. The distributions $\mathcal{F}_0=\partial_{u^1}\in\mathcal{V}(S_0)^\perp$, $\mathcal{F}_1=\partial_{x^3}\in\mathcal{V}(S_1)^\perp$ and $\mathcal{F}_2=\partial_{x^2}+u^2\partial_{x^1}\in\mathcal{V}(S_2)^\perp$ were used and the flat outputs $y^1=x^1-u^2x^2$ and $y^2=x^4$ follow at once by applying a coordinate

transformation based on the flow of $\mathcal{F}_2$ and regarding $\mathcal{F}_0$ and $\mathcal{F}_1$ no coordinate transformation is at need, since u^1 and x^3 are already non-derivative variables.

Remark 21 *Also in this example it holds true that* $S_0^{(1)} \subset S_1$ *and* $S_1^{(1)} \subset S_2$ *which enables one to construct the solutions based on the necessary condition (17) very easily.*

However, a different possible solution for $S_1 \oplus S_{1,c}$ can be based on choosing the distribution $\mathcal{F}_0 = \{\partial_{u^1}, \partial_{u^2}\}$ together with $S_1 = S_0^{(1)} = \{\omega_1^1, \omega_1^2\}$

$$\begin{aligned}\omega_1^1 &= \mathrm{d}x^1 - x^3\mathrm{d}x^4 - x^2\mathrm{d}t \\ \omega_1^2 &= \mathrm{d}x^2 - x^1\mathrm{d}x^4 - x^3\mathrm{d}t\end{aligned}$$

and $S_{1,c} = \{\mathrm{d}x^3 - (u^1 + x^2u^2)\mathrm{d}t\,, \mathrm{d}x^4 - u^2\mathrm{d}t\}$, where obviously $S_{1,c}$ is parameterizable with respect to u^1 and u^2. This choice for $\mathcal{F}_0$ and S_1 however leads to an 'dead end' since for S_1 the necessary condition (17) does not lead to a splitting $S_1 = S_2 \oplus S_{2,c}$. Therefore this choice of splitting has to be dismissed and one has to start again with S_0 (iteration).

6 Discussion

We have characterized a suitable normal form for 1-flat systems, which is in implicit triangular shape, see (10) that possesses the properties as in proposition 8 based on exterior algebra. Furthermore, we have discussed a constructive calculation scheme to transform 1-flat systems into that desired form. It should be mentioned again that we only provide sufficient conditions for a system to be 1-flat and that the constructive algorithm is in general not unique, and iterations might be necessary. Nevertheless, we believe that the presented normal-form is of interest in the analysis of the flatness problem, and our examples show that this implicit triangular form can be achieved by successive coordinate transformations in a rather straightforward manner. Additionally the well known Brunovsky form for systems that are linearizable by static feedback is naturally included in our approach, based on proposition 14.

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